Chapter 1

Decomposition in a Singularly Perturbed System

1.1 Integral Manifolds and Separation Movements

Consider the problem of separation singularly perturbed equations of the dynamics of the controlled system

$$\dot{x} = A_1(t)x + A_2(t)z + B_1(t)u + f_1(t),$$
$$\mu\dot{z} = A_3(t)x + A_4(t)z + B_2(t)u + f_2(t),$$
$$x(t_0) = x^0, \quad z(t_0) = z^0$$

or

$$\dot{y}(t,\mu) = A(t,\mu)y(t,\mu) + B(t,\mu)u(t) + f(t,\mu), \qquad (1.1.1)$$
$$y(t_i) = y^i, \quad i = 0,1,$$

where

$$A(t,\mu) = \begin{pmatrix} A_1(t) & A_2(t) \\ \frac{1}{\mu}A_3(t) & \frac{1}{\mu}A_4(t) \end{pmatrix}, \; y(t,\mu) = \begin{pmatrix} x(t,\mu) \\ z(t,\mu) \end{pmatrix}, \; B(t,\mu) = \begin{pmatrix} B_1(t) \\ \frac{B_2(t)}{\mu} \end{pmatrix},$$

$f(t,\mu) = \begin{pmatrix} f_1(t) \\ \frac{1}{\mu} f_2(t) \end{pmatrix}$, $x(t) \in R^n$, $z(t) \in R^m$ - vectors of slow and fast coordinate system (1.1.1) $u(t) \in R^r$ - control vector function; μ - a small positive parameter, $t \in [t_0, t_1]$, vector functions $f_1(t) \in R^n$, $f_2(t) \in R^m$ - characterize a constant external force.

Suppose that the following conditions are met:

I. Matrices $A_1(t)$, $A_2(t)$, $A_3(t)$, $A_4(t)-$ defined uniformly bounded and uniformly continuous with their derivatives with $t \in [t_0, t_1]$.

II. Eigenvalues value of matrix $A_4(t)$ submits to an inequality

$$\operatorname{Re}\lambda_i(t) \le -\gamma < 0, \quad (i=\overline{1,m}), \tag{1.1.2}$$

where $\gamma > 0$ - some a constant, $t \in [t_0, t_1]$.

At $u = 0$, instead of (1.1.1) we receive uncontrollable system

$$\begin{aligned} \dot{x} &= A_1(t)x + A_2(t)z + f_1(t), \\ \mu\dot{z} &= A_3(t)x + A_4(t)z + f_2(t). \end{aligned} \tag{1.1.3}$$

If the conditions I, II, then the system (1.2.2) is an integral manifold [98]

$$z = H(t,\mu)x + \tilde{z}\ . \tag{1.1.4}$$

Then the slow movement on the integral manifold (1.1.4) describes the system

$$\dot{x} = (A_1(t) + A_2(t)H(t,\mu))x + A_2(t)\tilde{z} + f_1(t), \tag{1.1.5}$$

where the matrix $H = H(t,\mu)$ and the vector $\tilde{z} = \tilde{z}(t,\mu)$ are determined from the equation

$$\begin{aligned} \mu\dot{H} &= -\mu H A_1(t) + A_4(t)H + A_3 - \mu H A_3(t)H, \\ \mu\dot{\tilde{z}} &= (A_4(t) - \mu H(t,\mu)A_2(t))\tilde{z} + (f_2(t) - \mu H(t,\mu) f_1(t)). \end{aligned} \tag{1.1.6}$$

Making the system (1.1.3) replacement $z = H(t,\mu)x + \tilde{z}(t,\mu) + \eta$ can be divided into fast and slow movements

$$\begin{aligned} \overline{x} &= (A_1(t) + A_2(t)H(t,\mu))x + A_2(t)\tilde{z} + f_1(t) + A_2(t)\eta, \\ \mu\dot{\eta} &= (A_4(t) - \mu H(t,\mu)A_2(t)\eta. \end{aligned} \tag{1.1.7}$$

This procedure is described in [44] when dividing of slow and fast movements uncontrollable system.

Direct application of this approach to system (1.1.1) is not possible. This is due to the following reasons: when considering problem of optimal control, we are interested primarily controllability of the system and the selection of the control function. A selection of the control function is carried out by various criteria of optimality and is associated with other problems.

Depending formulation of the problem, the above method can be applied only for the intermediate results of the general problem, as is done in [44].

If we act in the same way as the way the system (1.1.1), then the second equation (1.1.6) will be even additional term, which contains the control function, which has not yet been determined.

In the first equation (1.1.6) contains a small nonlinearity, it is necessary to set the initial condition in the beginning it is necessary to establish the existence and uniqueness of the solution of this equation. If the initial value problem for this equation is solvable, then there is another question that relates to the passage to the limit $\mu \to 0$ in the area of the boundary layer [18].

Given these observations outlined here offer significantly modified approach integral manifold, which allows for "a complete separation of" slow and fast coordinate system (1.1.1) and get a new system with traffic separation, which has all the properties of the original system. And this in turn makes it possible to formulate the optimal control problem under the constraint (1.1.1) in the form of the problem of moment [42], which is a new step towards the studied system in theory of optimal control.

We introduce the change of variables:

$$z(t,\mu) = \tilde{z}(t,\mu) + Hx(t,\mu), \quad (1.1.8)$$

$$x(t,\mu) = \tilde{x}(t,\mu) - \mu N\tilde{z}(t,\mu), \quad (1.1.9)$$

where the matrices $H = H(t,\mu) \leftrightarrow N = N(t,\mu)$ have dimensions respectively $m\times n,\ n\times m$ and will be determined by the parameters of the system (1.1.1). Later they called a matrix integral manifold.

From (1.1.8) and (1.1.9) we have the relation:

$$\begin{pmatrix} \tilde{x} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} E_n - \mu NH & \mu N \\ -H & E_m \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}, \tag{1.1.10}$$

$$\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} E_n & -\mu N \\ H & E_m - \mu HN \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{z} \end{pmatrix}. \tag{1.1.11}$$

We denote

$$M = M(t,\mu) = \begin{pmatrix} E_n & -\mu N \\ H & E_m - \mu HN \end{pmatrix}, \tag{1.1.12}$$

then

$$M^{-1} = \begin{pmatrix} E_n - \mu NH & \mu N \\ -H & E_m \end{pmatrix}. \tag{1.1.13}$$

It is easy to verify that for known H and N, $M \cdot M^{-1} = E_{n+m}$.

Since $y = \begin{pmatrix} x \\ z \end{pmatrix}$, $\tilde{y} = \begin{pmatrix} \tilde{x} \\ \tilde{z} \end{pmatrix}$, the ratio (1.1.10) and (1.1.11) are written as

$$\tilde{y} = M^{-1} \cdot y, \qquad y = M \cdot \tilde{y}. \tag{1.1.14}$$

For non-stationary matrices H and N depend on t and μ. Then the second relation (1.1.14) we have

$$\dot{y} = \dot{M} \cdot \tilde{y} + M \cdot \dot{\tilde{y}}. \tag{1.1.15}$$

In view of (1.1.14) and (1.1.15), the system (1.1.1) is transformed as

$$\dot{\tilde{y}} = (M^{-1}AM - M^{-1}\dot{M})\tilde{y} + M^{-1}Bu + M^{-1}f \ . \qquad (1.1.16)$$

We write the equation (1.1.16) in expanded form

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{z}} \end{pmatrix} = \begin{pmatrix} \tilde{A}_1(t,\mu) + N(-\mu H\tilde{A}_1(t,\mu) + A_3 + A_4H - \mu\dot{H}) & \mu\dot{N} - \mu\tilde{A}_1(t,\mu)N + N\tilde{A}_4(t,\mu) + A_2 + \mu N(\mu\dot{H} + \mu H\tilde{A}_1(t,\mu) - A_3 - A_4H)N \\ \frac{1}{\mu}(-\mu\dot{H} - \mu\tilde{A}_1(t,\mu) + A_3 + A_4H) & \frac{1}{\mu}\tilde{A}_4(t,\mu) + (\mu\dot{H} + \mu H\tilde{A}_1(t,\mu) - A_3 - A_4H)N \end{pmatrix}\begin{pmatrix} \tilde{x} \\ \tilde{z} \end{pmatrix} +$$

$$+\begin{pmatrix} \tilde{B}_1(t,\mu) \\ \frac{1}{\mu}\tilde{B}_2(t,\mu) \end{pmatrix}u + \begin{pmatrix} \tilde{f}_1(t,\mu) \\ \frac{1}{\mu}\tilde{f}_2(t,\mu) \end{pmatrix}, \qquad (1.1.17)$$

where

$$\tilde{A}_1(t,\mu) = A_1(t) + A_2(t)H(t,\mu), \qquad \tilde{A}_4(t,\mu) = A_4(t) - \mu H(t,\mu)A_2(t),$$

$$\tilde{B}_1(t,\mu) = B_1(t) + N(t,\mu)\tilde{B}_2(t,\mu), \qquad \tilde{B}_2(t,\mu) = B_2(t) - \mu H(t,\mu)B_1(t),$$

$$\tilde{f}_1(t,\mu) = f_1(t) + N(t,\mu)\tilde{f}_2(t,\mu), \quad \tilde{f}_2(t,\mu) = f_2(t) - \mu H(t,\mu)f_1(t), \quad (1.1.18)$$

$$u = u(t,\mu).$$

In order to slow and fast state variables of the system (1.1.1) divided, we require the following conditions:

$$-\mu\dot{H} - \mu H\tilde{A}_1(t,\mu) + A_3 + A_4H = 0 \ \text{ and}$$

$$\mu\dot{N} - \mu\tilde{A}_1(t,\mu)N + N\tilde{A}_4(t,\mu) + A_2 = 0 \, . \qquad (1.1.19)$$

Hence we have the following matrix equations from which we can determine the matrices-functions H and N:

$$\mu\dot{H} = -\mu HA_0 + A_3 + A_4H - \mu HA_2\left(H + A_4^{-1}A_3\right), \qquad (1.1.20)$$

$$\mu\dot{N} = \mu A_0 N - NA_4 + \mu NHA_2 + \mu A_2\left(H + A_4^{-1}A_3\right)N - A_2\,, \qquad (1.1.21)$$

where $A_0 = A_1 - A_2 A_4^{-1} A_3$.

If H and N satisfy the equations (1.1.20) and (1.1.21) then from (1.1.17) we have

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{z}} \end{pmatrix} = \begin{pmatrix} \tilde{A}_1(t,\mu) & 0 \\ 0 & \dfrac{1}{\mu}\tilde{A}_4(t,\mu) \end{pmatrix}\begin{pmatrix} \tilde{x} \\ \tilde{z} \end{pmatrix} + \begin{pmatrix} \tilde{B}_1(t,\mu) \\ \dfrac{1}{\mu}\tilde{B}_2(t,\mu) \end{pmatrix} u + \begin{pmatrix} \tilde{f}_1(t,\varpi) \\ \dfrac{1}{\mu}\tilde{f}_2(t,\mu) \end{pmatrix}$$

and

$$\dot{\tilde{x}} = \tilde{A}_1(t,\mu)\tilde{x} + \tilde{B}_1(t,\mu)u + \tilde{f}_1(t,\mu)\,, \qquad (1.1.22)$$

$$\mu\dot{\tilde{z}} = \tilde{A}_4(t,\mu)\tilde{z} + \tilde{B}_2(t,\mu)u + \tilde{f}_2(t,\mu)\,. \qquad (1.1.23)$$

The boundary conditions for equations (1.1.22) and (1.1.23) are defined as

$$\tilde{x}\left(t_0\right) = \tilde{x}^0, \quad \tilde{x}\left(t_1\right) = \tilde{x}^1, \quad \tilde{z}\left(t_0\right) = \tilde{z}^0, \quad \tilde{z}\left(t_1\right) = \tilde{z}^1, \qquad (1.1.24)$$

where $\tilde{x}^i = x^i + \mu N(t_i)\tilde{z}^i$, $\tilde{z}^i = z^i - H(t_i)x^i$, $i = 0, \quad 1$.

Equation (1.1.22), (1.1.23) and the boundary conditions (1.1.24) can be rewritten as

$$\dot{\tilde{y}} = \tilde{A}(t,\mu)\tilde{y} + \tilde{B}(t,\mu)u + \tilde{f}(t,\mu)\,, \ \tilde{y}(t_i) = \tilde{y}^i, \qquad i = 0, 1, \qquad (1.1.25)$$

where

$$\tilde{y} = M^{-1}y = \begin{pmatrix} \tilde{x} \\ \tilde{z} \end{pmatrix}, \quad y = \begin{pmatrix} x \\ z \end{pmatrix}, \quad \tilde{B}(t,\mu) = M^{-1}B = \begin{pmatrix} \tilde{B}_1(t,\mu) \\ \dfrac{1}{\mu}\tilde{B}_2(t,\mu) \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ \dfrac{1}{\mu}B_2 \end{pmatrix}, \qquad (1.1.26)$$

$\tilde{f}(t,\mu) = M^{-1} f = \begin{pmatrix} \tilde{f}_1(t,\mu) \\ \frac{1}{\mu}\tilde{f}_2(t,\mu) \end{pmatrix}$, $\tilde{x}$ and $\tilde{z}$ are determined from (1.1.8) and (1.1.9), and M^{-1} of (1.1.13),

$$\tilde{A}(t,\mu) = \begin{pmatrix} \tilde{A}_1(t,\mu) & 0 \\ 0 & \dfrac{\tilde{A}_4(t,\mu)}{\mu} \end{pmatrix}, \qquad (1.1.27)$$

where $\tilde{A}_1(t,\mu)$ and $\tilde{A}_4(t,\mu)$ are determined by the relations of (1.1.18).

As a result, we have the following result as a theorem.

Theorem 1.1.1. Suppose that the conditions I, II and differentiable matrices functions $H(t,\mu)$, $N(t,\mu)$ are satisfying the equation (1.1.20) and (1.1.21). Then the system (1.1.1) may be divided into two subsystems of lower order, respectively, which contain slow and fast coordinate system, where they are connected only in the control.

Thus, if the conditions of theorem 1.1.1, we obtain a new system with traffic separation, which is equivalent to the original system, as it has all the properties (controllability and stabilizability) of the original system (1.1.1).

Therefore, in the study problems of control as constraints can take differential constraints (1.1.22), (1.1.23).

1.2 Matrix of Integral Manifolds

Here we study the equation (1.1.20), (1.1.21) from which are determined by the matrices of integral manifold. We prove the existence and uniqueness of solutions of the equation (1.1.20). Show more mutually conjugate two equations that correspond to linear homogeneous parts of (1.1.20) and (1.1.21).

Theorem 1.2.1. If $\Phi_*(t_1 t_0)$ is transition matrix for the equation $\dot{p}(t) = -A_0'(t)p(t)$, and $\Psi(t_1 t_0, \mu)$ for the equation $\mu\dot{z}(t,\mu) = A_4(t)z(t,\mu)$, equation (1.1.20) with the initial condition $H(t_0,\mu) = H_0, \quad (H_0 \in G, G \subset R^{m\times n}$ - bounded set) at $\mu > 0$ equivalent to the integral equation.

$$H(t,\mu) = \Psi(t,t_0,\mu)H_0\Phi_*'(t,t_0) + \frac{1}{\mu}\int_{t_0}^{t}\Psi(t,s,\mu)A_3(s)\Phi_*'(t,s)ds -$$

$$-\int_{t_0}^{t}\Psi(t,s,\mu)H(s,\mu)A_2(s)\left(H(s,\mu) + A_4^{-1}(s)A_3(s)\right)\Phi_*'(\text{t,s})\text{ds} = \Psi(t,t_0,\mu)H_0\Phi_*'(t,t_0) +$$

$$+\frac{1}{\mu}\int_{t_0}^{t}\Psi(t,s,\mu)\Big(A_3(s) - \mu H(s,\mu)A_2(s)\left(H(s,\mu) + A_4^{-1}(s)A_3(s)\right)\Big)\Phi_*'(\text{t,s})\text{ds}, \qquad (1.2.1)$$

$t \in [t_0, t_1]$.

Proof. Differentiating (1.2.1) with respect to t and using the equation:

$$\dot{\Phi}_*(t,t_0) = -A_0'(t)\Phi_*(t,t_0), \qquad (1.2.2)$$

$$\mu\dot{\Psi}(t,t_0,\mu) = A_4(t)\Psi(t,t_0,\mu), \qquad (1.2.3)$$

We obtain

$$\dot{H}(t,\mu) = \frac{1}{\mu}A_4(t)\Psi(t,t_0,\mu)H_0\Phi_*'(t,t_0) - \Psi(t,t_0,\mu)H_0\Phi_*'(t,t_0)A_0(t) +$$

$$+\frac{1}{\mu^2}A_4(t)\int_{t_0}^{t}\Psi(t,s,\mu)\Big(A_3(s) - \mu H(s,\mu)A_2(s)\left(H(s,\mu) + A_4^{-1}(s)A_3(s)\right)\Big)\Phi_*'(t,s)ds -$$

$$-\frac{1}{\mu}\int_{t_0}^{t}\Psi(t,s,\mu)\Big(A_3(s) - \mu H(s,\mu)A_2(s)\left(H(s,\mu) + A_4^{-1}(s)A_3(s)\right)\Big)\Phi_*'(t,s)dsA_0(t) +$$

$$+\frac{1}{\mu}A_3(t) - H(t,\mu)A_2(t)\left(H(t,\mu) + A_4^{-1}(t)A_3(t)\right) = \frac{1}{\mu}A_4(t)[\Psi(t,t_0,\mu)H_0\Phi_*'(t,t_0) +$$

$$+\frac{1}{\mu}\int_{t_0}^{t}\Psi(t,s,\mu)\left(A_3(s)-\mu H(s,\mu)A_2(s)\left(H(s,\mu)+A_4^{-1}(s)A_3(s)\right)\right)\Phi_*'(t,s)ds]-$$

$$-[\Psi(t,t_0,\mu)H_0\Phi_*'(t,t_0)+\frac{1}{\mu}\int_{t_0}^{t}\Psi(t,s,\mu)\left(A_3(s)-\mu H(s,\mu)A_2(s)\left(H(s,\mu)+A_4^{-1}(s)A_3(s)\right)\right)\Phi_*'(t,s)ds]A_0(t)+$$

$$+\frac{1}{\mu}A_3(t)+H(t,\mu)A_2(t)\left(H(t,\mu)+A_4^{-1}(t)A_3(t)\right)=\frac{1}{\mu}A_4(t)H(t,\mu)-H(t,\mu)A_0(t)+$$

$$+\frac{1}{\mu}A_3(t)-H(t,\mu)A_2(t)\left(H(t,\mu)+A_4^{-1}(t)A_3(t)\right)..$$

Hence we have the equation (1.1.20), Q.E.D.

We now show that (1.1.20) (or (1.2.1)) has a unique solution when $0<\mu\le\mu_0<1$, where μ_0 - a positive constant. We introduce the notation:

$$H_0(t,\mu)=\Psi(t,t_0,\mu)H_0\Phi_*'(t,t_0)+\frac{1}{\mu}\int_{t_0}^{t}\Psi(t,s,\mu)A_3(s)\Phi_*'(t,s)ds\,, \quad (1.2.4)$$

$$K(H,t,\mu)=-\frac{1}{\mu}\int_{t_0}^{t}\Psi(t,s,\mu)H(s,\mu)A_2(s)\left(H(s,\mu)+A_4^{-1}(s)A_3(s)\right)\Phi_*'(t,s)ds\,. \quad (1.2.5)$$

Then the integral equation (1.2.1) is written as an operator equation

$$H(t,\mu)=H_0(t,\mu)+\mu K(H,t,\mu)\,, \quad (1.2.6)$$

where $K(H,t,\mu)$ - integral operator in the form (1.2.5).

At $\mu>0,\quad H_0\in G,\quad t\in[t_0,t_1]$ introduce the following notation:

$$M_1=\left\|H_0\right\|,\ M_2=\max_{t_0\le t\le t_1}\left\|A_2(t)\right\|,\ M_3=\max_{t_0\le t\le t_1}\left\|A_3(t)\right\|,$$

$$M_4=\max_{t_0\le t\le t_1}\left\|A_4^{-1}(t)\right\|\ M=\left\{M_1,\quad M_2,\quad M_3\quad M_4\right\}, \quad (1.2.7)$$

because by condition I and II of the matrices $A_2(t),\quad A_3(t)$ are uniformly bounded and $A_4(t)$ - stable matrix with $t\in[t_0,t_1]$, then we have

$$\left\|\Psi(t,s,\mu)\right\| \le C_1 e^{-\frac{\lambda_1(t-s)}{\mu}}, \quad \left\|\Phi'_*(\text{t,s})\right\| \le C_2 e^{\lambda_2(t-s)}, \tag{1.2.8}$$

$$\left(0 \le s \le t \le t_1, \quad 0 < \mu < \mu_0, \quad \lambda_1 > 0, \quad \lambda_2 > 0\right).$$

$$\left\|H_0(t,\mu)\right\| \le CMe^{-\frac{\lambda(t-t_0)}{\mu}} + \frac{MC}{\lambda}\left(1 - e^{-\frac{\lambda(t-t_0)}{\mu}}\right) \le \left(1 - \frac{1}{\lambda}\right)MCe^{-\frac{\lambda(t-t_0)}{\mu}} + \frac{MC}{\lambda} \le m_1, \tag{1.2.9}$$

where $m_1 > 0, \quad \lambda = \lambda_1\left(1 - \frac{\mu\lambda_2}{\lambda_1}\right) > 0, \; C = C_1 \cdot C_2, \quad \mu < \mu_1, \quad \mu_1 = \frac{\lambda_1}{\lambda_2}$.

At

$$\left\|H(t,\mu)\right\| \le m_2, \tag{1.2.10}$$

$$\left\|K(H,t,\mu)\right\| \le m_2\left(1 - e^{-\frac{\lambda(t-t_0)}{\mu}}\right)\left(m_2 + M^2\right) \le M^*. \tag{1.2.11}$$

Choose a number μ_2 such, that for any $\mu \le \mu_2$ there were

$$\left\|H_0(t,\mu) + \mu K(H,t,\mu)\right\| \le m_2.$$

For this it suffices choose

$$\mu_2 \le \min\left\{\mu_1, \frac{m_2 - m_1}{M^*}\right\}. \tag{1.2.12}$$

We construct a successive approximation $H_0, H_1, \ldots, H_k, \ldots$ by the formula

$$H_{k+1}(t,\mu) = H_0(t,\mu) + \mu K(H_k,t,\mu), \quad k = 0,1,2,\ldots \tag{1.2.13}$$

If $\left\|H_k(t,\mu)\right\| \le m_2$ at $t \in [t_0,t_1]$, then $H_{k+1}(t,\mu)$ is a continuous matrix function defined on $\left[t_0,t_1\right]$ and satisfying

$$\left\|H_{k+1}(t,\mu)\right\| \le \left\|H_0(t,\mu)\right\| + \mu\left\|K(H_k,t,\mu)\right\| \le m_1 + \mu M^* \le m_2. \tag{1.2.14}$$

For k=0 and k=1 have the inequality (1.2.14). By induction it holds for all $k \geq 0$.

Let the matrix $A_i(t)$, $(i=\overline{1,4})$ of the system (1.1.1) are defined, uniformly bounded and uniformly continuous together with its derivatives on $[t_0, t_1]$. As a $K(H,t,\mu)$ continuous function of its arguments, one can show that there exists a positive number L for any matrices functions $\overline{H}$ and $\overline{\overline{H}}$, satisfying the inequalities

$\left\|\overline{H}(t,\mu)\right\| \leq m_2$, $\left\|\overline{\overline{H}}(t,\mu)\right\| \leq m_2$, such that

$$\left\|K(\overline{H},t,\mu) - K(\overline{\overline{H}},t,\mu)\right\| \leq L\left\|\overline{H} - \overline{\overline{H}}\right\| \qquad (1.2.15)$$

when $\mu \leq \mu_2$. Subtract term by term from (1.2.13) with k=n-1 is the same equation for k=n-2. Then we obtain

$$H_n(t,\mu) - H_{n-1}(t,\mu) = \mu[K(H_{n-1},t,\mu) - K(H_{n-2},t,\mu)], \qquad (1.2.16)$$

in view of (1.2.15) from (1.2.16), we have

$$\left\|H_n(t,\mu) - H_{n-1}(t,\mu)\right\| \leq \mu L \max_{t_0 \leq t \leq t_1}\left\|H_{n-1}(t) - H_{n-2}(t)\right\|. \qquad (1.2.17)$$

We introduce the notation

$$r_n = \max_{t_0 \leq t \leq t_1}\left\|H_n(t,\mu) - H_{n-1}(t,\mu)\right\|, \qquad (1.2.18)$$

then from (1.2.17) we have

$$r_n \leq \mu L r_{n-1}. \qquad (1.2.19)$$

From the recurrence relation (1.2.17) we obtain

$$r_n \leq \mu L^{n-1} r_1, \qquad (1.2.20)$$

where $r_1 = \max\limits_{t_0 \le t \le t_1} \left\| H_1(t,\mu) - H_0(t,\mu) \right\| \le \mu \max\limits_{\substack{t_0 \le t \le t_1 \\ \|H\| \le m_2,\ \ \mu<\mu_2}} \left\| K(H,t,\mu) \right\| = \mu M^*$.

From (1.2.20) follows that the above series converges uniformly for $t \in [t_0, t_1]$ any choice $\mu < \mu_3$, and where $\mu_3 L = 1$.

Now we will put $\mu^* = \min\{\mu_2, \mu_3\}$, then at $\mu < \mu^*$ creation of successive approximations (1.2.13) it is possible to, and the sequence $\{H_k(t,\mu)\},\quad k = 0,1,\ldots$ converges uniformly on the interval $[t_0, t_1]$. Limit of this sequence satisfies (1.2.6) (or (1.2.1)).

The uniqueness of the solution follows from Gronwall's lemma, for any two solutions $\bar{H}$ и $\bar{\bar{H}}$ in the common domain of definition is valid assessment

$$\left\| \bar{H}(t,\mu) - \bar{\bar{H}}(t,\mu) \right\| \le \mu \left\| K(\bar{H},t,\mu) - K(\bar{\bar{H}},t,\mu) \right\| \le L \left\| \bar{H}(t,\mu) - \bar{\bar{H}}(t,\mu) \right\|.,$$

Is possible only if $\bar{H}(t,\mu) \equiv \bar{\bar{H}}(t,\mu)$.

Consider the matrix differential equations, which correspond to linear homogeneous parts of the equation (1.1.20), (1.1.21):

$$\mu \dot{\bar{H}} = -\mu \bar{H} A_0(t) + A_4(t) \bar{H}\ , \qquad (1.2.21)$$

$$\mu \dot{\bar{N}} = \mu A_0(t) \bar{N} - \bar{N} A_4(t)\,. \qquad (1.2.22)$$

where $\bar{H} = \bar{H}(t,\mu),\quad \bar{N} = \bar{N}(t,\mu),\quad t \in [t_0, t_1],\quad \bar{H} \in R^{m\times n}, \bar{N} \in R^{n\times m}$.

In the space $R^{m\times n}$ dot product [4]

$$\left(\bar{H}, \bar{N}\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} \bar{h}_{ij} \bar{n}_{ji} = Sp(\bar{H} \cdot \bar{N})\,. \qquad (1.2.23)$$

We show that the equation (1.2.22) will be paired for a homogeneous matrix equation (1.2.21). Indeed, if the equation (1.2.22) is conjugate to (1.2.21), its solution $\bar{N}(t)$ for any t satisfies $\left(\bar{H}(t), \bar{N}(t)\right) = Sp\left(\bar{H}(t) \cdot \bar{N}(t)\right) = const$, where the $\bar{H}(t)$ - solution of the equation (1.2.21).

On the basis of this definition we have

$$0 = \frac{d}{dt} Sp\left(\mu \bar{H}(t) \cdot \bar{N}(t)\right) = Sp\left(\mu \dot{\bar{H}}(t)\bar{N}(t) + \mu \bar{H}(t)\dot{\bar{N}}(t)\right) =$$
$$= Sp\left(-\mu \bar{H}(t)A_0(t) \cdot \bar{N}(t) + A_4(t)\bar{H}(t)\bar{N}(t) + \mu \bar{H}(t)\dot{\bar{N}}(t)\right).$$

Given the properties of the trace of the matrix, we obtain

$$0 = Sp\left\{\bar{H}(t)[\mu \dot{\bar{N}}(t) - \mu A_0(t)\bar{N}(t) + \bar{N}(t)A_4(t)]\right\}.$$

This condition must be executed under any $\bar{H}(t), \quad \bar{N}(t)$.

Then $\mu \dot{\bar{N}}(t) - \mu A_0(t)\bar{N}(t) + \bar{N}(t)A_4(t) = 0$.

Hence we have the equation (1.2.22). Now we write the equation (1.1.21) in the form

$$\mu \dot{N}(t) = \mu A_0(t)N(t) - N(t)A_4(t) +$$
$$+\mu\left[A_2(t)\left(H(t) + A_4^{-1}(t)A_3(t)\right)N(t) + N(t)H(t)A_2(t)\right] - A_2(t). \quad (1.2.24)$$

Let $H(t), \quad (t_0 \le t \le t_1)$ - solution of the equation (1.1.20). Then, based on the basic property of the adjoint equation (it is the solution of the original differential equation backward in time), the equation (1.2.24) can be written as the integral equation

$$N(t) = \Phi(t,t_1)N_1\Psi'_*(t,t_1,\mu) + \int_{t_1}^{t} \Phi(t,s)\left[A_2(s)\left(H(s) + A_4^{-1}(s)A_3(s)\right)N(s) + N(s)H(s)A_2(s)\right]\Psi'_*(t,s,\mu)ds -$$
$$-\frac{1}{\mu}\int_{t_1}^{t} \Phi(t,s)A_2(s)\Psi'_*(t,s,\mu)ds, \quad (1.2.25)$$

where $N_1 = N(t_1)$, $\Phi\,(t, t_1)$, $\Psi_*(t_0, t_1, \mu)$ are the transition matrix for the equation $\dot{x}(t) = A_0(t)x(t)$, $\mu\dot{g}(t,\mu) = -A_4'(t)g(t,)$.

The existence and uniqueness of solutions of the equation (1.2.25) can be proved similarly to the previous case.

It should be noted that the differential equation (1.2.22), the boundary condition is not specified in the initial moment of time t_0, but in the end of the transition process. This follows from the basic properties of the ad joint equation.

1.3 Matrix Transition of Singularly Perturbed System and Its Asymptotic Behavior

Consider the question of constructing a transition matrix of the system (1.1.1). Suppose that the conditions I and II. Then for sufficiently small values of the parameter, the transition matrix $Y(t, t_0, \mu)$ of the system

$$\dot{y}(t,\mu) = A(t,\mu)y(t,\mu)\,,\ \ y(t_0) = y^0\,, \tag{1.3.1}$$

corresponding to the system (1.1.1) can be determined as the solution of the matrix differential equation

$$\dot{Y}(t, t_0, \mu) = A(t,\mu)Y(t, t_0, \mu), \quad Y(t_0, t_0, \mu) = E_{n+m}\ . \tag{1.3.2}$$

Matrix $Y(t, t_0, \mu)$ divided into blocks

$$Y(t, t_0, \mu) = \begin{pmatrix} Y_1(t, t_0, \mu) & \mu Y_2(t, t_0, \mu) \\ Y_3(t, t_0, \mu) & \mu Y_4(t, t_0 \mu) \end{pmatrix}, \tag{1.3.3}$$

in view of (1.3.3) from the (1.3.2) we have:

$$\dot{Y}_1 = A_1(t)Y_1 + A_2(t)Y_3, \quad Y_1(t_0, t_0, \mu) = E_n\,, \tag{1.3.4}$$

$$\mu\dot{Y}_3 = A_3(t)Y_1 + A_4(t)Y_3\,,\ \ Y_3(t_0, t_0, \mu) = 0\,,$$

$$\dot{Y}_2 = A_1(t)Y_2 + A_2(t)Y_4\,,\ Y_2(t_0, t_0, \mu) = 0\,, \tag{1.3.5}$$

$$\mu\dot{Y}_4 = A_3(t)Y_2 + A_4(t)Y_4\,,\ Y_4(t_0, t_0, \mu) = \frac{1}{\mu}E_m\,.$$

Solution of the Cauchy problem (1.3.4) and (1.3.5) will be determined in the form [17]:

$$Y_i(t, t_0, \mu) = \overline{Y}_i(t, t_0, \mu) + \Pi Y_i(\tau, \mu), \quad i = 1, 2, 3, 4\,, \tag{1.3.6}$$

where

$$\overline{Y}_i(t, t_0, \mu) = \sum_{k=0}^{\infty} Y_{ik}(t, t_0)\mu^k, \quad i = 1, 2, 3, 4\,, \tag{1.3.7}$$

$$\Pi Y_i(\tau, \mu) = \sum_{k=0}^{\infty} \Pi_k Y_i(\tau)\mu^k, i = 1, 2, 3\,,$$

$$\Pi Y_4(\tau, \mu) = \frac{1}{\mu}\Pi_{-1}Y_4(\tau) + \sum_{k=0}^{\infty} \Pi_k Y_4(\tau)\mu^k\,, \quad \tau = \frac{t - t_0}{\mu}\,,$$

$$\Pi Y_4(\tau, \mu) = \frac{1}{\mu}\Pi_{-1}Y_4(\tau) + \sum_{k=0}^{\infty} \Pi_k Y_4(\tau)\mu^k\,, \quad \tau = \frac{t - t_0}{\mu}\,.$$

We substitute (1.3.6) to (1.3.4) and (1.3.5). Further in these systems after function replacement $\overline{\mathrm{Y}}_i \cdot (t, t_0, \mu)$, $\Pi \mathrm{Y}_i(t, \mu)$ decomposition (1.3.7), equating the coefficients of like powers, and separately depending on t, and separately depending on the τ, we obtain the equation for determining the terms of the expansion (1.3.7).

At μ^{-1} with regard τ

$$\frac{d\Pi_{-1}Y_4(\tau)}{d\tau} = A_{40}(t_0)\Pi_{-1}Y_4(\tau)\,,\ \Pi_{-1}Y_4(0) = E_m\,. \tag{1.3.8}$$

At μ^0 with regard t:

$$\frac{d\bar{Y}_{10}(t,t_0)}{dt}=(A_1-A_2A_4^{-1}A_3)\bar{Y}_{10}(t,t_0),\;\; Y_{10}(t_0,t_0)=E_n\,, \tag{1.3.9}$$

$$\bar{Y}_{30}(t,t_0)=-A_4^{-1}A_3\bar{Y}_{10}(t,t_0)\,,$$

$$\frac{d\bar{Y}_{20}(t,t_0)}{dt}=(A_1-A_2A_4^{-1}A_3)\bar{Y}_{20}(t,t_0)\,,\; \bar{Y}_{20}(t_0,t_0)=-A_{20}(t_0)A_{40}^{-1}(t_0)\,, \tag{1.3.10}$$

$$\bar{Y}_{40}(t\;\,,t_0)=-A_4^{-1}A_3\bar{Y}_{20}(t,t_0)\,.$$

At μ^0 with regard τ :

$$\frac{d\Pi_0Y_1(\tau)}{d\tau}=0,\;\; \frac{d\Pi_0Y_3(\tau)}{d\tau}=A_{30}(t_0)\Pi_0Y_1(\tau)+A_{40}(t_0)\Pi_0Y_3(\tau)\,, \tag{1.3.11}$$

$$\frac{d\Pi_0Y_2(\tau)}{d\tau}=A_{20}(t_0)\Pi_{-1}Y_4(\tau), \tag{1.3.12}$$

$$\frac{d\Pi_0Y_4(\tau)}{d\tau}=A_{30}(t_0)\Pi_0Y_2(\tau)+A_{40}(t_0)\Pi_0Y_4(\tau)+A_{41}(t_0)\Pi_{-1}Y_4\,,$$

where $A_{i0}(t_0)=A_i(t_0),i=1,2,3,4$.

For the (1.3.8) - (1.3.12) we have the initial conditions:

$$\bar{Y}_{10}(t_0,t_0)+\Pi_0Y_1(0)=E_n,\;\; \bar{Y}_{30}(t_0,t_0)+\Pi_0Y_3(0)=0\,, \tag{1.3.13}$$

$$\bar{Y}_{20}(t_0,t_0)+\Pi_0Y_2(0)=0,\quad \bar{Y}_{40}(t_0,t_0)+\Pi_0Y_4(0)=0\,.$$

The solution of equation (1.3.8) written in the form

$$\Pi_{-1}Y_4(\tau)=\exp\left(A_4(t_0)\tau\right). \tag{1.3.14}$$

Solutions of system (1.3.9) - (1.3.12) with the initial conditions (1.3.13) are given by:

$$\Pi_0Y_1(\tau)=0\,,\; \Pi_0Y_2(\tau)=A_{20}(t_0)A_{40}^{-1}(t_0)\exp\left(A_{40}(t_0)\tau\right), \tag{1.3.15}$$

$$\Pi_0Y_3(\tau)=\exp\left(A_{40}(t_0)\tau\right)A_4^{-1}(t_0)A_3(t_0)\,,$$

$$\Pi_0 Y_4(\tau) = -\exp\left(A_4(t_0)\tau\right) A_{40}^{-1}(t_0) A_{30}(t_0) A_{20}(t_0) A_{40}^{-1}(t_0) +$$

$$+\int_0^{\tau} \exp\left(A_4(t_0)(\tau - s)\right)\left(A_{41}(t_0) + A_{30}(t_0) A_{20}(t_0) A_{40}^{-1}(t_0)\right)\exp\left(A_{40}(t_0)s\right) ds,$$

$$\overline{Y}_{10}(t,t_0) = \exp\left(\int_{t_0}^{t}\left(A_1(\sigma) - A_2(\sigma)A_4^{-1}(\sigma)A_3(\sigma)\right)d\sigma\right), \qquad (1.3.16)$$

$$\overline{Y}_{30}(t\ ,t_0) = -A_4^{-1}(t_0)A_3(t_0)\overline{Y}_{10}(t,t_0)\ ,$$

$$\overline{Y}_{20}(t,t_0) = -\exp\left(\int_{t_0}^{t}\left(A_1(\sigma) - A_2(\sigma)A_4^{-1}(\sigma)A_3(\sigma)\right)d\sigma\right)\cdot A_{20}(t_0)A_{40}^{-1}(t_0)\ ,$$

$$\overline{Y}_{40}(t\ ,t_0) = -A_4^{-1}(t_0)A_3(t_0)\overline{Y}_{20}(t,t_0)\ .$$

For the matrix $\Pi_0 Y_i(\tau)$, i=2,3,4 the following estimated holds on the interval $(0,\infty)$ [18]:

$$\left\|\Pi_0 Y_i(\tau)\right\| \le C\cdot\exp(-\beta\tau), \quad (\tau \ge 0, \quad \beta > 0, \quad i = 2,3,4)\ . \qquad (1.3.17)$$

Proceeding similarly with the terms of the expansion of the order μ^p can be with respect to t write the following:

$$\frac{d\overline{Y}_{1p}}{dt} = A_1(t)\overline{Y}_{1p}(t,t_0) + A_2(t)\overline{Y}_{3p}(t,t_0), \qquad (1.3.18)$$

$$\frac{d\overline{Y}_{3p}}{dt} = A_3(t)\overline{Y}_{1p}(t,t_0) + A_4(t)\overline{Y}_{3p}(t,t_0)\ ,$$

$$\frac{d\overline{Y}_{2p}}{dt} = A_1(t)\overline{Y}_{2p}(t,t_0) + A_2(t)\overline{Y}_{4p}(t,t_0),$$

$\dfrac{d\overline{Y}_{4,p-1}}{dt} = A_3(t)\overline{Y}_{2p}(t,t_0) + A_4(t)\overline{Y}_{4p}(t,t_0)$, with regard τ :

$$\frac{d\Pi_p Y_1(\tau)}{d\tau} = d_{1p}(\tau), \quad \frac{d\Pi_p Y_3(\tau)}{d\tau} = A_{30}(t_0)\Pi_p Y_1(\tau) + A_{40}(t_0)\Pi_p Y_3(\tau) + d_{3p}(\tau), \quad (1.3.19)$$

$$\frac{d\Pi_p Y_2(\tau)}{d\tau} = d_{2p}(\tau), \quad \frac{d\Pi_p Y_4(\tau)}{d\tau} = A_{30}(t_0)\Pi_p Y_2(\tau) + A_{40}(t_0)\Pi_p Y_4(\tau) + d_{4p}(\tau),$$

where $d_{1p}(\tau) = \sum_{j=0}^{p-1} A_{1,p-1-j}(t_0)\Pi_j Y_1(\tau) + \sum_{j=0}^{p-1} A_{2,p-1-j}(t_0)\Pi_j Y_3(\tau)$,

$$d_{2p}(\tau) = A_{2p}(t_0)\Pi_{-1}Y_4(\tau) + \sum_{j=0}^{p-1} A_{1,p-1-j}(t_0)\Pi_j Y_2(\tau) + \sum_{j=0}^{p-1} A_{2,p-j}(t_0)\Pi_j Y_4(\tau),$$

$$d_{3p}(\tau) = \sum_{j=0}^{p-1} A_{3,p-j}(t_0)\Pi_j Y_1(\tau) + \sum_{j=0}^{p-1} A_{4,p-j}(t_0)\Pi_j Y_3(\tau),$$

$$d_{4p}(\tau) = A_{4p+1}(t_0)\Pi_{-1}Y_4(\tau) + \sum_{j=0}^{p-1} A_{3,p-j}(t_0)\Pi_j Y_2(\tau) + \sum_{j=0}^{p-1} A_{4,p-j}(t_0)\Pi_j Y_4(\tau).$$

The initial conditions for the equation (1.3.18), (1.3.19) is determined by the condition:

$$\overline{Y}_{1p}(t_0, t_0) + \Pi_p Y_1(0) = 0, \quad \overline{Y}_{2p}(t_0, t_0) + \Pi_p Y_2(0) = 0, \quad (1.3.20)$$

$$\overline{Y}_{3p}(t_0, t_0) + \Pi_p Y_3(0) = 0, \quad \overline{Y}_{4p}(t_0, t_0) + \Pi_p Y_4(0) = 0.$$

Suppose now that defined terms in the expansions (1.3.7) up to order p inclusive, i.e. obtained the following partial sums

$$Y_{ip}(t, t_0, \mu) = \sum_{j=0}^{p} \left[\overline{Y}_{ij}(t, t_0) + \Pi_j Y_i(\tau)\right]\mu^j, \quad i = 1, 2, 3, \quad (1.3.21)$$

$$Y_{4p}(t, t_0, \mu) = \frac{1}{\mu}\Pi_{-1}Y_4(\tau) + \sum_{j=0}^{p} \left[\overline{Y}_{4j}(t, t_0) + \Pi_j Y_4(\tau)\right]\mu^j.$$

Then solutions of systems of equations (1.3.4), (1.3.5) may be represented as

$$Y_i(t, t_0, \mu) = Y_{ip}(t, t_0, \mu) + \phi_i(t, \mu), \quad i = 1, 2, 3, 4. \quad (1.3.22)$$

Thus, if the conditions I and II, that for sufficiently small values of the parameter μ for matrix functions $\phi_i(t,\mu)$ and $\Pi_j Y_i(\tau)$ will we have the estimates:

$$\left\|\phi_i(t,\mu)\right\| \le C\mu^{p+1},\ \ i=1,2,3,4,\quad t\in[t_0,t_1], \tag{1.3.23}$$

$$\left\|\Pi_j Y_i(\tau)\right\| \le C\exp(-\beta\tau),\quad \tau\ge 0,$$

where C, β -const.

Now from the decomposition (1.3.7) select terms which form the zero-order approximation of the transition matrix $Y(t,t_0\mu)$. Zero approximation is denoted by $Y_0(t,t_0,\mu)$. Consider the degenerate system, which is obtained from (1.3.1) at $\mu=0$

$$\dot{\bar{x}} = A_1(t)\bar{x} + A_2(t)\bar{z},\ \ 0 = A_3(t)\bar{x} + A_4(t)\bar{z}\,. \tag{1.3.24}$$

If after $x(t,\mu)=\bar{x}(t,\mu)+\Pi x(\tau,\mu)$ and $z(t,\mu)=\bar{z}(t,\mu)+\Pi z(\tau,\mu)$ denote the solution of the Cauchy problem (1.3.1), and through $x^0(t)$ and $z^0(t)$ solution of the degenerate system (1.3.24) provided $x(t_0)=x^0$, then the conditions of theorem Tikhonov, we have

$$x(t,\mu)=x^0(t)+O(\mu),\ \ z(t,\mu)=z^0(t)+\frac{1}{\mu}\Pi_{-1}z(\tau)+O(\mu), \tag{1.3.25}$$

where $t\in\left[t_0,t_1\right]$, $\dfrac{1}{\mu}\Pi_{-1}z(\tau)$ - the first term of the expansion frontier function $\Pi z(\tau)$. Applying this provision to the problem (1.3.2), we can write the zero approximation $Y_0(t,t_0,\mu)$ transition matrix $Y(t,t_0\mu)$, which gives a uniform asymptotic accuracy $O(\mu)$ at all of interest to us time interval

$$Y_0(t,t_0,\mu)=\begin{pmatrix} \exp\left(\int_{t_0}^{t} A_0(s)ds\right) & 0 \\ -A_4^{-1}(t_0)A_3(t_0)\exp\left(\int_{t_0}^{t} A_0(s)ds\right)+\exp\left(A_4(t_0)\frac{t-t_0}{\mu}\right)A_4^{-1}(t_0)A_3(t_0) & \exp\left(A_4(t_0)\frac{t-t_0}{\mu}\right) \end{pmatrix}.$$

The continuity of the matrix $A_i(t)$, $i=1,2,3,4$ on the interval $[t_0,t_1]$ follows that at each point $t\in[t_0,t_1]$ derivative of the function $M(t)=\int_{t_0}^{t} A_0(\lambda)d\lambda$ the upper limit is equal to the integrand function $A_0(t)$, i.e. $\frac{dM(t)}{dt}=A_0(t)$.

We have the following

Theorem 1.3.1. Matrix $Y_0(t,t_0,\mu)$ is the solution of the matrix differential equation

$$\dot{Y}_0(t,t_0,\mu)=\dot{A}_*(t)Y_0(t,t_0,\mu),\ \ Y_0(t_0,t_0)=E_{n+m}\,, \tag{1.3.26}$$

where $A_*(t,\mu)=\begin{pmatrix} A_0(t) & 0 \\ -A_4^{-1}(t_0)A_3(t_0)A_0(t)+\frac{1}{\mu}A_3(t_0) & \frac{1}{\mu}A_4(t_0) \end{pmatrix}$.

Proof. We represent the matrix $Y_0(t,t_0,\mu)$ in the form

$$Y_0(t,t_0,\mu)=V\overline{Y}_0(t,t_0,\mu)V^{-1}, \tag{1.3.27}$$

where $V=\begin{pmatrix} E_n & 0 \\ -A_4^{-1}(t_0)A_3(t_0) & E_m \end{pmatrix}$, $\overline{Y}_0(t,t_0,\mu)=\begin{pmatrix} e^{\int_{t_0}^{t} A_0(\lambda)d\lambda} & 0 \\ 0 & e^{A_4(t_0)\left(\frac{t-t_0}{\mu}\right)} \end{pmatrix}$.

We introduced the notation $\overline{A}(t,\mu)=\begin{pmatrix} A_0(t) & 0 \\ 0 & \frac{A_4(t_0)}{\mu} \end{pmatrix}$.

Then differentiating both sides of (1.3.27), we obtain

$$\frac{dY(t,t_0,\mu)}{dt} = V\frac{d\overline{Y_0}(t,t_0,\mu)}{dt}V^{-1} = V\overline{A}(t,\mu)\overline{Y_0}(t,t_0,\mu)V^{-1} = V\overline{A}(t,\mu)V^{-1}V\overline{Y}_0(t,t_0,\mu)V^{-1} =$$
$$= V\overline{A}(t,\mu)V^{-1}Y_0(t,t_0,\mu) = A_*(t,\mu)Y_0(t,t_0,\mu).$$

At $t = t_0$ $Y_0(t_0,t_0) = E_{n+m}$, Q.E.D.

Theorem 1.3.2. If the condition (1.1.2), then the matrix $A_*(t,\mu)$ defines a system for which there exists an integral manifold

$$\overline{z} = -A_4^{-1}(t_0)A_3(t_0)\overline{x} + \tilde{z}. \tag{1.3.28}$$

The movement, which is described by the system

$$\dot{\overline{x}}(t) = A_0(t)\overline{x}(t),\;\; \overline{x}(t_0) = x^0, \tag{1.3.29}$$

$$\mu\dot{\tilde{z}}(t) = A_4(t_0)\tilde{z}(t),\;\; \tilde{z}(t_0) = \tilde{z}^0,$$

where $\tilde{z}^0 = A_4^{-1}(t_0)A_3(t_0)x^0 + z^0$.

In this case, there is a limit relation $\lim\limits_{\mu\to 0}\tilde{z}(t,\mu) = 0$ or $\lim\limits_{\mu\to 0}\overline{z}(t,\mu) = -A_4^{-1}(t_0)A_3(t_0)\overline{x}^0(t) = \tilde{z}^0(t)$.

Proof.

Indeed, the system $\dot{\overline{y}}(t) = A_*(t,\mu)\overline{y}(t)$, $\overline{y}(t_0) = y^0$, where $y(t) = col(\overline{x}(t), \overline{z}(t))$ in expanded form is written as

$$\dot{\overline{x}}(t) = A_0(t)\overline{x}(t),\;\; \overline{x}(t_0) = x^0, \tag{1.3.30}$$

$$\dot{\overline{z}}(t) = -A_4^{-1}(t_0)A_3(t_0)A_0(t)\overline{x}(t) + \frac{1}{\mu}A_3(t_0)\overline{x}(t) + \frac{1}{\mu}A_4(t_0)\overline{z}(t),\;\; \overline{z}(t_0) = z^0$$

In view of (1.3.28) from the second equation of (1.3.30), we obtain

$\mu\dot{\tilde{z}}(t) = \mu\left(-A_4^{-1}(t_0)A_3(t_0)\dot{\bar{x}}(t) + \dot{\bar{z}}(t)\right) = -\mu A_4^{-1}(t_0)A_3(t_0)A_0(t)\bar{x}(t) + A_3(t_0)\bar{x}(t) - A_3(t_0)x(t) + A_4(t_0)\tilde{z}$ or $\mu\dot{\tilde{z}}(t) = A_4(t_0)\tilde{z}(t),\quad \tilde{z}(t_0) = \tilde{z}^0,$ where $\tilde{z}^0 = A_4^{-1}(t_0)A_3(t_0)x^0 + z^0$. This, together with the first equation (1.3.30) gives the system (1.3.29) and its solution can be written as:

$$\bar{x}^0(t) = \exp(\int_{t_0}^{t} A_0(\lambda)dx)x^0,\ \ \tilde{z}^0(t,\mu) = \exp(A_4(t_0)(t-t_0)/\mu)\tilde{z}^0 \text{ or}$$

$$\bar{z}^0(t,\mu) = -A_4^{-1}(t_0)A_3(t_0)\bar{x}^0(t) + \exp(A_4(t_0)(t-t_0)/\mu).$$

By hypothesis, the eigenvalues $\lambda_i(t_0)$ matrix $A_4(t_0)$ satisfy the inequality $\operatorname{Re}\lambda_i(t_0) < -\gamma < 0$ and $\mu \to 0$ we get the specified limit relations. At $\mu << 1$ will have the representation $\bar{z}^0(t,\mu) = \bar{z}^0(t) + O(e^{-\gamma\frac{(t-t_0)}{\mu}}),\quad \gamma > 0$.

1.4 Converting Matrix Transition on the Integral Manifold

We now state and prove a theorem which gives a formula of the transition matrix of the system (1.2.1) and allows you to split the state vector of the system to slow and fast components.

Theorem 1.4.1. Let the matrices $\Phi(t,s,\mu)$ and $\Psi(t,s,\mu)$ are transition matrices of homogeneous systems $\dot{\tilde{x}} = \tilde{A}_1\tilde{x}$, $\mu\cdot\dot{\tilde{z}} = \tilde{A}_4\cdot\tilde{z}$ i.e. they satisfy the equations

$$\dot{\Phi}(t,s,\mu) = \tilde{A}_1(t,\mu)\Phi(t,s,\mu),\ \ \Phi(s,s,\mu) = E_n, \tag{1.4.1}$$

$$\mu\dot{\Psi}(t,s,\mu) = \tilde{A}_4(t,\mu)\Psi(t,s,\mu),\ \ \Psi(s,s,\mu) = E_m/\mu, \tag{1.4.2}$$

where matrices $\tilde{A}_1(t,\mu)$ and $\tilde{A}_4(t,\mu)$ are determined from (1.1.18). Then the transition matrix $Y(t,s,\mu)$ systems (1.3.1), the corresponding system (1.1.1), can be represented as

$$Y(t,s,\mu) = M(t,\mu)G(t,s,\mu)M^{-1}(s,\mu)\,, \tag{1.4.3}$$

where matrices $M(t,\mu)$ and $M^{-1}(t,\mu)$ are determined from (1.1.12) and (1.1.13), respectively;

$$G(t,s,\mu) = diag\left(\Phi(t,s,\mu), \quad \Psi(t,s,\mu)\right), \tag{1.4.4}$$

Matrix $\Phi(t,s,\mu)$ and $\Psi(t,s,\mu)$ will be called the transition matrix of slow and fast subsystems (1.3.1).

Proof. Matrix $Y(t,s,\mu)$ we will divide into blocks in the form of

$$Y\left(t,s,\mu\right) = \begin{pmatrix} Y_1\left(t,s,\mu\right) & \mu Y\left(t,s,\mu\right) \\ Y_3\left(t,s,\mu\right) & \mu Y\left(t,s,\mu\right) \end{pmatrix}. \tag{1.4.5}$$

Then from the equation

$$\dot{Y}\left(t,s,\mu\right) = A\left(t,\mu\right)Y\left(t,s,\mu\right),\ Y\left(t,s,\mu\right) = E_{m+m} \tag{1.4.6}$$

we have

$$\dot{Y}_1 = A_1\left(t\right)Y_1 + A_2\left(t\right)Y_3\,,\ Y\left(s,s,\mu\right) = E_m\,,\ \mu\dot{Y}_3 = A_3\left(t\right)Y_1 + A_4\left(t\right)Y_3\,,$$
$$Y_3\left(s,s,\mu\right) = 0 \tag{1.4.7}$$

$$\dot{Y}_2 = A_1\left(t\right)Y_2 + A_2\left(t\right)Y_4\,,\ Y_2\left(s,s,\mu\right) = E_m\,,\ \mu\dot{Y}_4 = A_3\left(t\right)Y_2 + A_4\left(t\right)Y_4\,,$$
$$Y_4\left(s,s,\mu\right) = \frac{1}{\mu}E_m \tag{1.4.8}$$

Now, in these systems, we make the change of variables

$$Y_3 = H(t,\mu)Y_1 + Z\,, \tag{1.4.9}$$

$$Y_4 = H(t,\mu)Y_2 + \Psi\,, \tag{1.4.10}$$

where $Y_1 = Y_1(t,s,\mu)$, $Y_2 = Y_2(t,s,\mu)$, $Y_3 = Y_3(t,s,\mu)$, $Y_4 = Y_4(t,s,\mu)$, $\Psi = \Psi(t,s,\mu)$.

$H(t,\mu)$, $Z=Z(t,s,\mu)$ - matrices of order $m\times n$, elements which are regularly depend on μ . Then the system (1.4.7) takes the form:

$$\dot{Y}_1 = A_1(t)Y_1 + A_2(t)H(t,\mu)Y_1 + A_2(t)Z = (A_1(t) + A_2(t)H(t,\mu))Y_1 + A_2(t)Z \ ,$$

$$\dot{Y}_1 = \tilde{A}_1(t,\mu)Y_1 + A_2(t)Z \ , \ Y_1(s,s,\mu) = E_n \ , \qquad (1.4.11)$$

$$\mu(\dot{H}(t,\mu)Y_1 + H(t,\mu)\dot{Y}_1 + \dot{Z}) = A_3(t)Y_1 + A_4(t)H(t,\mu)Y_1 + A_4(t)Z,$$

$$(\mu\dot{H}(t,\mu) + \mu H(t,\mu)\tilde{A}_1(t,\mu))Y_1 + \mu H(t,\mu)A_2(t)Z + \mu\dot{Z} = (A_3(t) + A_4(t)H(t,\mu))Y_1 + A_4(t)Z \ ,$$

$$(\mu\dot{H}(t,\mu) + \mu H(t,\mu)\tilde{A}_1(t,\mu))Y_1 + \mu\dot{Z} = (A_3(t) + A_4(t)H(t,\mu))Y_1 + (A_4(t) - \mu H(t,\mu)A_2(t))Z.$$

In view of (1.1.19), we obtain from the last equation

$$\mu\dot{Z} = \tilde{A}_4(t,\mu)Z, \ \ Z(s,s,\mu) = -H(s,\mu) \ . \qquad (1.4.12)$$

Substituting (1.4.10) into the first equation (1.4.8), we obtain

$$\dot{Y}_2 = \tilde{A}_1(t,\mu)Y_2 + A_2(t)\Psi, \ \ Y_2(s,s,\mu) = 0 \ . \qquad (1.4.13)$$

Differentiating function (1.4.10) over the t obtain $\dot{Y}_4 = \dot{H}(t,\mu)Y_2 + H(t,\mu)\dot{Y}_2 + \dot{\Psi}$. The meaning Y_4 substitute into the second equation (1.4.8)

$$\mu\dot{H}(t,\mu)Y_2 + \mu H(t,\mu)\dot{Y}_2 + \mu\dot{\Psi} = A_3(t)Y_2 + A_4(t)H(t,\mu)Y_2 + A_4(t)\Psi,$$

$$\mu\dot{H}(t,\mu)Y_2 + \mu H(t,\mu)\tilde{A}_1(t,\mu)Y_2 + \mu H(t,\mu)A_2(t)\Psi + \mu\dot{\Psi} =$$
$$(A_3(t) + A_4(t)H(t,\mu))Y_2 + A_4(t)\Psi \ .$$

Given the ratio from (1.1.19) as a result we have an equation with respect Ψ

$$\mu\dot{\Psi} = \tilde{A}_4(t,\mu)\Psi, \quad \Psi(s,s,\mu) = \frac{1}{\mu}E_m \ . \qquad (1.4.14)$$

Now we eliminate Z from the equation (1.4.11). For this we introduce the matrix

$F=F(t,s,\mu)$ considering that the matrix $H(t,\mu)$ already known

$$F=Y_1+\mu N(t,\mu)Z\ . \qquad (1.4.15)$$

Then the systems (1.4.11) and (1.4.12) take the form:

$$\dot{F}=\tilde{A}_1(t,\mu)F\ ,\ F(s,s,\mu)=E_n-\mu N(s,\mu)H(s,\mu)\,, \qquad (1.4.16)$$

$$\mu\dot{Z}=\tilde{A}_4(t,\mu)Z\,,\ Z(s,s,\mu)=-H(s,\mu)\,, \qquad (1.4.17)$$

where matrix $N(t,\mu)$ is determined from the second equation (1.1.19).

We now show that the matrices F , Z and Y_2 are defined in terms of matrices $\Phi(t,s,\mu)$ and $\Psi(t,s,\mu)$. By hypothesis 1.4.1 the matrices Φ and Ψ satisfy equations (1.4.1) and (1.4.2), then the matrices:

$$F=\Phi(E_n-\mu N(s,\mu)H(s,\mu))\,, \qquad (1.4.18)$$

$$Z=-\Psi H(s,\mu)\,, \qquad (1.4.19)$$

$$Y_2=\int_s^t\Phi(t,\sigma,\mu)A_2(\sigma)\Psi(\sigma,s,\mu)d\sigma\,, \qquad (1.4.20)$$

satisfy the equations (1.4.16), (1.4.17) and (1.4.13), respectively. This statement for the first two relations is easily verified. We present the following lemma.

Lemma. Let the matrix $N(t,\mu)$ $\left(t\in[t_0,t_1],\quad \mu>0\right)$ is the solution of the second equation in (1.1.19). Then

$$\frac{1}{\mu}\int_s^t\Phi(t,\sigma,\mu)A_2(\sigma)\Psi(\sigma,s,\mu)d\sigma=\Phi(t,s,\mu)N(s,\mu)-N(t,\mu)\Psi(t,s,\mu)\,, \qquad (1.4.21)$$

Proof. Equation (1.4.21) can be written in the form

$$\frac{1}{\mu}\int_s^t\Phi(t,\sigma,\mu)A_2(\sigma)\Psi(\sigma,s,\mu)d\sigma=-\Phi(t,\sigma,\mu)N(\sigma,\mu)\Psi(\sigma,s,\mu)\Big|_s^t \qquad (1.4.22)$$

Then from (1.4.22) obtain

$$\frac{d\left(\Phi(t,\sigma,\mu)N(\sigma,\mu)\Psi(\sigma,s,\mu)\right)}{d\sigma}=-\frac{1}{\mu}\Phi(t,\sigma,\mu)A_2(\sigma)\Psi(\sigma,s,\mu).$$

Using one of the properties of the transition matrix, hence we have

$$-\tilde{A}_1(\sigma,\mu)\Phi(t,\sigma,\mu)N(\sigma,\mu)\Psi(\sigma,s,\mu)+\Phi(t,\sigma,\mu)\frac{d\left(N(\sigma,\mu)\right)}{d\sigma}\Psi(\sigma,s,\mu)+$$
$$+\frac{1}{\mu}\Phi(t,\sigma,\mu)N(\sigma,\mu)\tilde{A}_4(\sigma,\mu)\Psi(\sigma,s,\mu)=-\frac{1}{\mu}\Phi(t,\sigma,\mu)A_2(\sigma)\Psi(\sigma,s,\mu),$$

$$-\tilde{A}(t,\mu)N(t,\mu)\Psi(t,s,\mu)+\dot{N}(t,\mu)\Psi(t,s,\mu)$$
$$+\frac{1}{\mu}\tilde{A}_4(t,\mu)\Psi(t,s,\mu)+\frac{1}{\mu}A_2(t)\Psi(t,s,\mu)=0$$

Multiplying this equality on the right by the matrix $\Phi(s,t,\mu)$ have

$$\mu\dot{N}(t,\mu)-\mu\tilde{A}_1(t,\mu)N(t,\mu)=-A_2(t)-N(t,\mu)\tilde{A}_4(t,\mu)$$

As a result, we obtain the second equation in (1.1.19), since by hypothesis matrix lemma $N(t,\mu)$ is the solution of the second equation (1.1.19). Therefore, the formula (1.4.21) is true. Q.E.D.

On the basis of the formula (1.4.21), the expression (1.4.20) can be written as

$$Y_2=\mu\left(\mathit{\Phi}N(s,\mu)-N(t,\mu)\Psi\right). \qquad (1.4.23)$$

Differentiating function (1.4.21) to the t and considering that $N(t,\mu)$ is the solution of the second equation (1.1.19), we obtain

$$\dot{Y}_2=\mu\tilde{A}_1(t,\mu)\Phi N(s,\mu)+N(t,\mu)\tilde{A}_4(t,\mu)\Psi-\mu\tilde{A}_1(t,\mu)N(t,\mu)\Psi$$
$$+A_2(t)\Psi-N(t,\mu)\tilde{A}_4(t,\mu)\Psi=$$

$$=\mu\tilde{A}_1(t,\mu)(\Phi N(s,\mu)-N(t,\mu)\Psi)+A_2(t)\Psi=\tilde{A}_1(t,\mu)Y_2+A_2(t)\Psi.$$

At $t = s$ rom the (1.4.20) it follows that $Y_2(s,s,\mu) = 0$.

From the relations (1.4.9), (1.4.10), (1.4.15), (1.4.18), (1.4.19), (1.4.21) we have: $Y_1 = \Phi(E_n - \mu N(s,\mu)H(s,\mu)) + \mu N(t,\mu)\Psi H(s,\mu)$,

$$Y_2 = \Phi N(s,\mu) - N(t,\mu)\Psi,$$

$$Y_3 = H(t,\mu)\Phi(E_n - \mu N(s,\mu)H(s,\mu)) - (E_m - \mu H(t,\mu)N(t,\mu))\Psi H(s,\mu), \quad (1.4.24)$$

$$Y_4 = H(t,\mu)\Phi N(s,\mu) + \frac{1}{\mu}(E_m - \mu H(t,\mu)N(s,\mu))\Psi.$$

Substituting the values of Y_i (i=1,2,3,4) from (1.4.24) the right-hand side of (1.4.5) we obtain (1.4.3). Q.E.D.

It should be noted that the system (1.2.1) may be replaced by an integral equation (Cauchy formula):

$$y(t,\mu) = Y(t,s,\mu)y\left(s,\mu\right) + \int_s^t Y(t,s,\mu)B(s,\mu)u(s,\mu)ds + \int_s^t Y(t,s,\mu)f(s,\mu)ds \,. \quad (1.4.25)$$

Using the relations (1.4.3), (1.1.26) from (1.4.25) can be easily obtained integral equation, which is equivalent to the differential equation (1.1.25)

$$\tilde{y}(t,\mu) = G(t,t_0,\mu)\tilde{y}(s,\mu) + \int_s^t G(t,\sigma,\mu)\tilde{B}(\sigma,\mu)u(\sigma,\mu)d\sigma + \int_s^t G(t,\sigma,\mu)\tilde{f}(\sigma,\mu)d\sigma \quad (1.4.26)$$

where $\tilde{B}(t,\mu) = M^{-1}(t,\mu)B(t,\mu)$, $\tilde{f}(t,\mu) = M^{-1}(t,\mu)f(t,\mu)$, $\tilde{y}^0 = \tilde{y}(t_0)$, is transition matrix $G(t,t_0,\mu)$ determined from (1.4.4).

Now suppose that the matrices $\Phi(t,t_0,\mu)$ and $\Psi(t,t_0,\mu)$ are transitional matrix of the system (1.1.22) and (1.1.23). Along with the system (1.1.22) and (1.1.23), we consider another system

$$\dot{\overline{x}} = A_0(t)\overline{x} + B_0(t)\overline{u} + f_0(t), \;\; \overline{x}(t_0) = \overline{x}^0, \;\; \overline{x}(t_1) = \overline{x}^1, \quad (1.4.27)$$

$$\mu \dot{\bar{z}}_* = A_4(t)\bar{z}_* + B_2(t)\bar{u} + f_2(t), \;\; \bar{z}_*(t_0) = \bar{z}_*^0, \;\; \bar{z}_*(t_1) = \bar{z}_*^1,$$

where

$$A_0(t) = A_1(t) - A_2(t)A_4^{-1}(t)A_3(t), \quad B_0(t) = B_1(t) - A_2(t)A_4^{-1}(t)B_2(t),$$

$$f_0(t) = f_1(t) - A_2(t)A_4^{-1}(t)f_2(t), \;\; \dot{z}_* = \bar{z} + A_4^{-1}A_3\bar{x}, \qquad (1.4.28)$$

$\bar{x}, \bar{z}$ - vectors of state variables of the degenerate system, which is obtained from (1.1.22) and (1.1.23) at $\mu = 0$. Have the following theorem.

Theorem 1.4.2. Let $A_0(t), \;\; A_4(t)$ - stable matrices and corresponding to them transition matrices satisfy the inequalities

$$\left\|\bar{\Phi}(t,t_0)\right\| \le c\exp\left(-m(t-t_0)\right), \;\; \left\|\bar{\Psi}(t,t_0,\mu)\right\| \le c\exp\left(-\gamma(t-t_0)/\mu\right). \quad (1.4.29)$$

Then at m>1, $0 < \mu < \mu_0 < 1$ and $t_0 \le t \le t_1$ the eigenvalues value matrices $\tilde{A}_1(t,\mu), \tilde{A}_4(t,\mu)$ will be «close» to the eigenvalues of the matrices $A_0(t), \; A_4(t)$, in the sense of negativity their real parts, where

$$\mu_0 = \min\left\{\frac{1}{d_2c}, \;\; \frac{\gamma}{d_1c}\right\}, \qquad (1.4.30)$$

$$d_1 = \max_{t_0 \le s \le \sigma \le t \le t_1} L_1\left\|A_2(\sigma)\right\|, \;\; d_2 = \max_{t_0 \le s \le \sigma \le t \le t_1} L_2\left\|A_2(\sigma)\right\|, \qquad (1.4.31)$$

At the same time we have the estimates:

$$\left\|\Phi(t,t_0,\mu)\right\| \le c\exp\left(-m(t-t_0)\right), \;\; \left\|\Psi(t,t_0,\mu)\right\| \le c\exp\left(-\gamma(t-t_0)/\mu\right), \quad (1.4.32)$$

where the $m_1 = m - 1, \;\; \gamma_1 = \gamma - \mu d_1 c, \;\; c, m, \gamma$ - positive constants,

$\bar{\Phi}(t,t_0), \bar{\Psi}(t,t_0,\mu)$ - transition matrices slow and fast subsystems (1.4.27).

Proof. Let

$$H(t,\mu) = -A_4^{-1}(t)A_3(t) + \mu h(t,\mu)\,, \qquad (1.4.33)$$

where the $h(t,\mu)$ satisfies the matrix equation

$$\mu\dot{h}(t,\mu) + \mu h(t,\mu)A_0^*(t,\mu) = A_3^*(t,\mu) + A_4^*(t,\mu)h(t,\mu),$$
$$A_0^*(t,\mu) = A_0(t) + \mu A_2(t)h(t,\mu),$$

$$A_3^*(t) = \frac{d}{dt}(A_4^{-1}(t)A_3(t)) + A_4^{-1}(t)A_3(t)A_0(t), \;\; A_4^*(t,\mu) = A_4(t) + \mu A_4^{-1}(t)A_3(t)A_2(t)\,.$$

In view of (1.4.33), matrices $\tilde{A}_1(t,\mu), \tilde{A}_4(t,\mu)$ are defined as

$$\tilde{A}_1(t,\mu) = A_0(t) + \mu A_2(t)h(t,\mu), \qquad (1.4.34)$$

$$\tilde{A}_4(t,\mu) = A_4(t) + \mu A_4^{-1}(t)A_3(t)A_2(t) - \mu^2 h(t,\mu)A_2(t,\mu).$$

As shown in 1.2, matrix $H(t,\mu)$ is the solution of the integral equation (1.2.1) and defined as the limit of a sequence of continuous functions in a closed interval $[t_0, t_1]$, then can specify the number of μ_1 such that $0 < \mu \le \mu_1$ in the interim $[t_0, t_1]$ there are limitations:

$$\|H(t,\mu)\| \le L_1, \;\; \|h(t,\mu)\| \le L_2, \qquad (1.4.35)$$

where the L_1, L_2 - positive numbers. Transition matrices $\Phi(t,t_0,\mu)$ and $\Psi(t,t_0,\mu)$ may be represented in the form:

$$\Phi(t,t_0,\mu) = \bar{\Phi}(t,t_0) + \mu\varphi(t,t_0,\mu), \;\; \Psi(t,t_0,\mu) = \bar{\Psi}(t,t_0,\mu) + \mu\eta(t,t_0,\mu), \qquad (1.4.36)$$

where the $\varphi(t,t_0,\mu)$, $\eta(t,t_0,\mu)$ - matrices functions, $\bar{\Phi}(t,t_0), \bar{\Psi}(t,t_0,\mu)$ - transition matrices slow and fast subsystems (1.4.27).

The process of determining the functions $\varphi(t,t_0,\mu)$ and $\eta(t,t_0,\mu)$ leads us to clarify the main question: for sufficiently small values of the parameter μ, the eigenvalues of matrices $\tilde{A}_1(t,\mu)$ and $\tilde{A}_4(t,\mu)$ will indeed be close to the eigenvalues of the matrix $A_0(t)$ and $A_4(t)$ respectively.

We assume that $A_0(t)$ and $A_4(t)$ - stable matrix. Then the corresponding transition matrices satisfy the inequalities (1.4.29). By assumption, the matrix $\bar{\Phi}(t,t_0)$ and $\bar{\Psi}(t,t_0,\mu)$ satisfy the equations:

$$\dot{\bar{\Phi}}(t,t_0) = A_0(t)\Phi(t,t_0), \ \dot{\Phi}(t_0,t_0) = E_n, \qquad (1.4.37)$$

$$\mu\dot{\bar{\Psi}}(t,t_0,\mu) = A_4(t)\bar{\Psi}(t,t_0,\mu), \ \bar{\Psi}(t_0,t_0,\mu) = E_m / \mu. \qquad (1.4.38)$$

Then, taking into account (1.4.33), (1.4.36) from the (1.4.1) and (1.4.2), obtain

$$\dot{\varphi}(t,t_0,\mu) = A_0(t)\varphi(t,t_0,\mu) + A_2(t)h(t,\mu)\left(\bar{\Phi}(t,t_0) + \mu\varphi(t,t_0,\mu)\right),$$
$$\varphi(t_0,t_0,\mu) = 0, \qquad (1.4.39)$$

$$\mu\dot{\eta}(t,t_0,\mu) = A_4(t)\eta(t,t_0,\mu) - H(t,\mu)A_2(t)\left(\bar{\Psi}(t,t_0,\mu) + \mu\eta(t,t_0,\mu)\right),$$
$$\eta(t_0,t_0,\mu) = 0. \qquad (1.4.40)$$

Equations (4.1.39) and (4.1.40) are equivalent to the following integral equation:

$$\varphi(t,t_0,\mu) = \int_{t_0}^{t} \bar{\Phi}(t,\sigma)A_2(\sigma)h(\sigma,\mu)\left(\bar{\Phi}(\sigma,t_0) + \mu\varphi(\sigma,t_0,\mu)\right)d\sigma, \qquad (1.4.41)$$

$$\eta(t,t_0,\mu) = \frac{1}{\mu}\int_{t_0}^{t} \bar{\Psi}(t,\sigma,\mu)H(\sigma,\mu)A_2(\sigma)\left(\bar{\Psi}(\sigma,t_0,\mu) + \mu\eta(\sigma,t_0,\mu)\right)d\sigma. \qquad (1.4.42)$$

We now define two sequences:

$$\varphi_0(t,t_0,\mu) = \int_{t_0}^{t} \bar{\Phi}(t,\sigma)A_2(\sigma)h(\sigma,\mu)\bar{\Phi}(\sigma,t_0)d\sigma, \qquad (1.4.43)$$

$$\varphi_k(t,t_0,\mu) = \varphi_0(t,t_0,\mu) + \mu\int_{t_0}^{t}\bar{\Phi}(t,\sigma)A_2(\sigma)h(\sigma,\mu)\varphi_{k-1}(\sigma,t_0,\mu)d\sigma,$$

$$\eta_0(t,t_0,\mu) = \frac{1}{\mu}\int_{t_0}^{t}\bar{\Psi}(t,\sigma,\mu)H(\sigma,\mu)A_2(\sigma)\bar{\Psi}(\sigma,t_0,\mu)d\sigma \qquad 1.4.44)$$

$$\eta_k(t,t_0,\mu) = \eta_0(t,t_0,\mu) + \int_{t_0}^{t}\bar{\Psi}(t,\sigma,\mu)H(\sigma,\mu)A_2(\sigma)\eta_{k-1}(\sigma,t_0,\mu)d\sigma.$$

Investigate the convergence of the sequences (1.4.43) and (4.1.44).

We introduce the notation in the form (1.4.31). Now we estimate the total members of the following series:

$$\varphi_0 + \sum_{n=1}^{\infty}\left(\varphi_n - \varphi_{n-1}\right),\ \ \eta_0 + \sum_{k=1}^{\infty}\left(\eta_n - \eta_{n-1}\right). \qquad (1.4.45)$$

In view of (1.4.31) from the (1.4.43) obtain

$$\|\varphi_0\| \le d_2c^2(t-t_0)\exp\left(-m(t-t_0)\right),$$
$$\|\varphi_1 - \varphi_0\| \le \mu d_2^2c^3\left((t-t_0)^2/2!\right)\exp\left(-m(t-t_0)\right).$$

By induction, we obtain the inequality

$$\|\varphi_n - \varphi_{n+1}\| \le \mu^n d_2^{n+1}c^{n+2}\left((t-t_0)^{n+1}/(n+1)!\right)\exp\left(-(t-t_0)\right).$$

Of these estimates imply that the first row of (1.4.45) converges at $\mu < 1/(d_2c)$, and evenly and as a majorant series supports a number of

$$d_2c^2\left(t-t_0+(t-t_0)/2!+(t-t_0)^3/3!+\ldots+(t-t_0)^n/n!+\ldots\right)\exp\left(-m(t-t_0)\right),$$

which is the sum of $d_2c^2\left(\exp\left(t-t_0\right)-1\right)\exp\left(-m(t-t_0)\right)$.

Consequently, there $\varphi(t,t_0)$ - limit of the first series (1.4.45) exists. Function $\phi(t,t_0)$ satisfies to the equation (1.4.41) and it satisfies the inequality

$$\|\varphi\| \le d_2 c^2 \left(\exp\left(t - t_0\right) - 1\right) \exp\left(-m(t - t_0)\right), \qquad (1.4.46)$$

At $\mu < 1/(d_2 c)$.

Similarly, from (1.4.37), we have $\|\eta_0\| \le (1/\mu) d_1 c^2 \exp\left(-(\gamma(t - t_0))/\mu\right)$ and $\|\eta_n - \eta_{n-1}\| \le (1/\mu) d_1^{n+1} c^{n+2} \left((t - t_0)^{n+1}/(n+1)!\right) \exp\left(-(\gamma(t - t_0)/\mu\right)$.

Then there $\eta(t, t_0, \mu)$ - limit of the second row (1.4.45) exists. The limit function $\eta(t, t_0, \mu)$ at $\mu > 0$ is a solution equation (1.4.44) and in this case we obtain the estimate

$$\|\eta\| \le (c/\mu)\left(\exp\left(d_1 c(t - t_0)\right) - 1\right) \exp\left(-\left(\gamma(t - t_0)\right)/\mu\right). \qquad (1.4.47)$$

Now, using (14.29) and (1.4.46), (1.4.47) from the (1.4.36), obtain (1.4.32).